5. The local multiplicity of a holomorphic map

It is proved in this Chapter that the algebraic multiplicity of a holomorphic map coincides with its geometric multiplicity, that is with the index of the singular point of the corresponding holomorphic field. Although this result was known classically it seems that a detailed proof was published only in the paper [139] of V. P. Palamodov. The idea of the elementary proof presented below is due to A. G. Kushnirenko [106].

The index of a singular point of a real vector field can be computed as the signature of an appropriate quadratic form on the local algebra of the singularity (the formula of Levine–Eisenbud–Khimshiashvili [62], [97]). We prove this formula, based on nondegenerate quadratic forms on local algebras (Grothendieck duality) with the help of an n-dimensional generalisation of the theorem of Abel on the trace of a holomorphic form.

5.1 Multiplicity

Let $f:(\mathbb{C}^n, a)\to(\mathbb{C}^n, 0)$ be a holomorphic map-germ at a point a. Consider the algebra $\mathbb{C}\{x\}_a$ of all holomorphic function-germs at a. The germs of the components of f generate an ideal $I_{f,a}$ in this algebra.

Definition: the *multiplicity* of the germ f at the point a is the dimension of its local algebra

$$\mu_a[f] = \dim_{\mathbb{C}} Q_{f,a}\ ;\ Q_{f,a} = \mathbb{C}\{x\}_a/I_{f,a}.$$

A germ is said to be *of finite multiplicity* if its multiplicity is finite.

Example 1: If f is a nondegenerate linear operator then its multiplicity at 0 is equal to 1.

Example 2: Let $f_1 = x_1x_2^2, f_2 = x_1^2 + x_2^3$. We associate to the monomial $x_1^{k_1}x_2^{k_2}$ the point (k_1, k_2) of the integral lattice (Fig. 37a). We then note the monomials

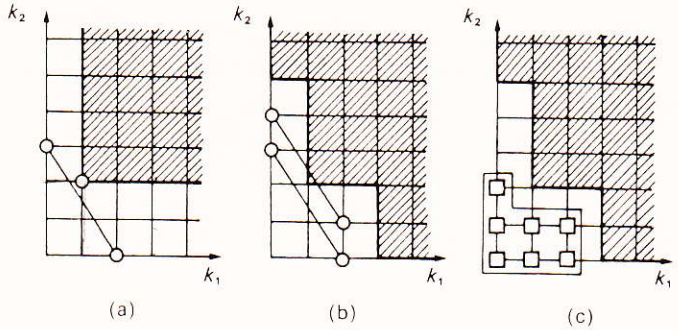

Fig. 37.

belonging to the ideal $I = (f_1, f_2)$. All the monomials in the hatched region of the figure belong to it together with $x_1x_2^2$. The binomial f_2 is formed by the segment with end-points (2, 0) and (0, 3). By moving this segment one place to the right we may convince ourselves that $x_1^3 \in I$ and by moving it up two places that $x_2^5 \in I$. Therefore all the monomials in the region hatched in Fig. 37b lie in I. In Fig. 37c seven monomials are indicated determining a $\mathbb{C}$-basis for the algebra $Q_{f,0}$. Thus $\mu_0[f] = 7$.

Example 3: Let $f_1 = x_2^2 - x_1x_2, f_2 = x_1x_2 - x_1^3$. Again we represent f_1 and f_2 by segments (Fig. 38a). The vertices of the zigzag path in Fig. 38a correspond to monomials which are congruent modulo the ideal $I = (f_1, f_2)$. Therefore $x_1x_2^2$ is congruent to monomials of arbitrarily high degree modulo I. It is not difficult to verify that $x_1x_2^2 \in I$ (for example, this is clear from the fact that $x_1x_2^2 \equiv x_1x_2^2 \cdot x_1$ mod I).

Therefore all the monomials in the region hatched in Fig. 38b lie in the ideal. A basis for $Q_{f,0}$ is generated by the five monomials enclosed in Fig. 38c, $\mu[f] = 5$.

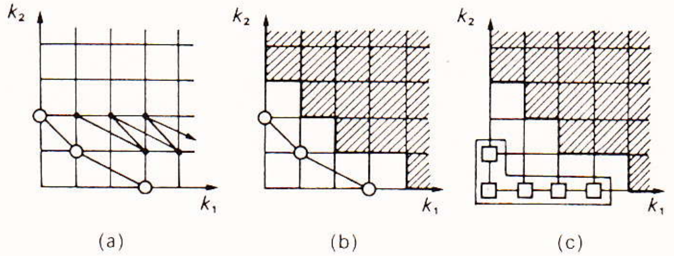

Fig. 38.

5.2 The index is equal to the multiplicity

Definition: The *index* $\mathrm{ind}_a[f]$ of a map-germ f at a point a is the degree of the map $f/\|f\|:S_\varepsilon^{2n-1}\to S_1^{2n-1}$ of a sufficiently small sphere $\|x-a\|=\varepsilon$ in the source space to the unit sphere in the image space.

If there is a neighbourhood of a in which there is no inverse image of 0 apart from possibly the point a itself then the index is well-defined (does not depend on the choice of the small sphere S_ε^{2n-1}). The index of a germ at a non-isolated zero is not defined. The multiplicity and index of a root a of a system of holomorphic equations $f_1=\ldots=f_n=0$, defined in a neighbourhood of a are just the multiplicity and index of the map-germ $f=(f_1,\ldots,f_n)$ at a.

Theorem 1: *The index of a holomorphic germ of finite multiplicity is equal to its multiplicity.*

Theorem 2: *A holomorphic map-germ fails to be of finite multiplicity at a point a, if and only if a is a non-isolated inverse image of zero of the germ.*

The proof of Theorem 2 is given in Section 5.9. The proof of Theorem 1 is given below. It is based on Propositions 1°–7°, formulated below and proved in Sections 5.3–5.8.

(1°) **The universality of the Pham map.**

Definition: The map $\Phi^m:\mathbb{C}^n\to\mathbb{C}^n$, defined by the formulas

$$y_1=x_1^{m_1},\ldots,y_n=x_n^{m_n},$$

is called the *Pham map*.

Definition: Two germs f and g at a point a are said to be *algebraically equivalent* or, briefly, *A-equivalent*, if there is a germ of a holomorphic family of linear nondegenerate maps $A(x)\in\mathrm{GL}(n,\mathbb{C})$ such that $f(x)=A(x)g(x)$.

Proposition: *Let $f:(\mathbb{C}^n, 0) \to \mathbb{C}^n$ be a map-germ of finite multiplicity. Then there exists a Pham map Φ^m such that f at 0 is A-equivalent to the map-germ $\Phi^m_\varepsilon = \Phi^m + \varepsilon f$ for arbitrary $\varepsilon \neq 0$.*

In other words, by a small deformation of a Pham map one can obtain any germ of finite multiplicity (up to A-equivalence).

(2°) **Proposition:** *The index and multiplicity at 0 of the Pham map coincide.*

(3°) **Proposition:** *The indices of A-equivalent germs are equal.*

(4°) **Proposition:** *The multiplicities of A-equivalent germs are equal.*

(5°) **Additivity of the index:** Let a system of n holomorphic equations in $\mathbb{C}^n$ depend holomorphically on a parameter.

Under changes of parameter a multiple root of the system may decompose.

Proposition: *The sum of the indices of the roots, formed by the decomposition of a multiple root of the system, is equal to the index of that root.*

(6°) **Subadditivity of the multiplicity.**

Proposition: *The sum of the multiplicities of the roots, formed by the decomposition of a multiple root of the system, does not exceed the multiplicity of that root.*

(7°) **Proposition:** *The multiplicity of a root is not less than its index.*

Proof of Theorem 1: Let $f:(\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ be a map-germ of finite multiplicity. Choose a Pham map Φ such that the germs f and $\Phi_\varepsilon = \Phi + \varepsilon f$ at zero are A-equivalent for $\varepsilon \neq 0$ (by 1°). Choose a sufficiently small neighbourhood U

of 0. Choose a sufficiently small $\varepsilon(U) > 0$. Consider the deformed Pham map $\Phi_\varepsilon = \Psi$. Let a_i be the roots of the system $\Psi = 0$, lying in the neighbourhood U. We obtain a chain of relations:

$$\mu_0[\Phi] \geqq \Sigma\mu_{a_i}[\Psi] \qquad \text{(by 6°)},$$

$$(1) \quad \mu_{a_i}[\Psi] \geqq \operatorname{ind}_{a_i}[\Psi] \qquad \text{(by 7°)},$$

$$\Sigma \operatorname{ind}_{a_i}[\Psi] = \operatorname{ind}_0[\Phi] \qquad \text{(by 5°)},$$

$$\operatorname{ind}_0[\Phi] = \mu_0[\Phi] \qquad \text{(by 2°)}.$$

From this chain it follows that all the inequalities in it are equalities. Since $f(0) = 0$, among the roots a_i is the point 0. Consequently

$$\mu_0[\Psi] = \operatorname{ind}_0[\Psi]$$

(since the inequality (1) has become an equality). But, since the germs f and Ψ are A-equivalent, we have

$$\mu_0[f] = \mu_0[\Psi] \qquad \text{(by 3°)},$$

$$\operatorname{ind}_0[f] = \operatorname{ind}_0[\Psi] \qquad \text{(by 4°)}.$$

Thus Theorem 1 has been proved in the case that $f(0) = 0$. On the other hand if $f(0) \neq 0$ then, as is easily proved,

$$\mu_0[f] = \operatorname{ind}_0[f] = 0.$$

5.3 The index of a real germ

The index is defined not only for holomorphic germs but also for smooth maps of real spaces.

Let $f:(\mathbb{R}^n, a) \to \mathbb{R}^n$ be a smooth germ at a point a.

Definition: The *index* $\operatorname{ind}_a[f]$ is the degree of the map $f/\|f\|: S_\varepsilon^{n-1} \to S_1^{n-1}$ of a sufficiently small sphere $\|x - a\| = \varepsilon$ in the source space to the unit sphere in the target space.

The index is not defined if a is a non-isolated zero of f.

Example: If $f(0) = 0$ and the Jacobian matrix of f at 0 is nondegenerate then the index of 0 is equal to plus or minus unity, depending on the sign of the Jacobian.

Suppose that in a closed ball $B \subset \mathbb{R}^n$ there are no zeros of the map $f: (\mathbb{R}^n, 0) \to \mathbb{R}^n$ except possibly the point 0 and let f_ε be an arbitrary smooth deformation of f.

Proposition 1: *For sufficiently small ε the sum of the indices of the zeros of the disturbed map f_ε in B is equal to the index of 0 of the original map f, provided that the number of these zeros is finite.*

In fact: (1°) All the maps $\varphi_\varepsilon = f_\varepsilon / \| f_\varepsilon \| : \partial B \to S_1^{n-1}$, for sufficiently small ε, are mutually homotopic. (2°) The degree of the map φ_ε is equal to the sum of the indices of the zeros of the map f_ε in the ball B.

Corollary: *The index of the point 0 of the map f is equal to the number of preimages in B of an arbitrary sufficiently small regular value $\varepsilon \in \mathbb{R}^n$, counted with the sign of the Jacobian at these points.*

For the proof it is sufficient to apply to the deformation $f_\varepsilon = f - \varepsilon$ the assertion of Proposition 1 and to use the computation of the index of a nondegenerate zero.

Definition: Two germs $f, g: (\mathbb{R}^n, 0) \to \mathbb{R}^n$ are said to be *real A-equivalent* if there is a germ of a smooth family of linear maps $A(x): \mathbb{R}^n \to \mathbb{R}^n$ such that $\det A(0) > 0$ and $g(x) = A(x) f(x)$.

Proposition 2: *The indices of real A-equivalent germs are equal.*

Proof: Since $\det A(0) > 0$, it is possible to join A with E by a homotopy A_t with $\det A_t(x) > 0$. The homotopy $g_t = A_t f$ joins g to f and has no zeros on the small sphere.

5.4 The index of a holomorphic germ

Proposition 1: *The determinant of the real form* $\hat{A}:\mathbb{R}^{2n}\to\mathbb{R}^{2n}$ *of a nondegenerate complex linear map* $A:\mathbb{C}^n\to\mathbb{C}^n$ *is positive.*

Proof: $\det\hat{A} = |\det A|^2$ (the formula is obtained by a direct computation with respect to a basis in which the matrix A has triangular form).

[A second proof: (1°) The set of nondegenerate linear operators $A:\mathbb{C}^n\to\mathbb{C}^n$ is connected. For the proof it is sufficient to join two nondegenerate matrices by a complex line; it intersects the set of degenerate matrices in not more than n points. (2°) Join a nondegenerate complex operator to 1 by a path consisting of nondegenerate complex operators. The real forms of these operators are nondegenerate (since nondegeneracy means invertibility). Consequently the determinants of all these real forms are positive.]

Corollary: *A-equivalent holomorphic germs have the same index.*

Proof: The real forms of holomorphic A-equivalent germs are real A-equivalent. In fact if $g = Af$ then the real form $\hat{g} = \hat{A}\hat{f}$ and $\det\hat{A}(0) > 0$.

Let B be a closed ball with centre at the point $a\in\mathbb{C}^n$. Suppose that the holomorphic map f is nowhere zero on $B\backslash a$.

Proposition 2: *The index at a of the germ of f is equal to the number of preimages in B of an arbitrary sufficiently small regular value ε.*

Proof: The index is equal to the number of preimages of ε, counted with signs of the Jacobian of f (see Section 5.2). According to the Lemma this sign is always positive.

Remark: Consider a holomorphic map of a 2n-dimensional compact domain in $\mathbb{C}^n$, not having a zero on the boundary of the region. Then *the degree of the map* $f/\|f\|$ *of the boundary to* S_1^{2n-1} *is non-negative* because this degree is equal to the number of preimages of ε.

Proposition 3: *Suppose that a map has no zeros on the boundary of a bounded domain $U \subset \mathbb{C}^n$ and that the degree of the map $g/\|g\|$ of the boundary of U to the unit sphere is equal to k. Then the system $g = 0$ has a finite number of roots in U and the sum of their indices is equal to k.*

Proposition 3 follows from the following Lemma.

Lemma: *Under the conditions of Proposition* 3 *the number of geometrically distinct solutions of the system $g = 0$ in U does not exceed k.*

Proof: Suppose that the system has $k + 1$ roots $a_1, \ldots, a_{k+1}$.
(1°) There exists a polynomial map $P: \mathbb{C}^n \to \mathbb{C}^n$, for which the points $a_1, \ldots, a_{k+1}$ are nondegenerate roots.
(2°) The map $g_\varepsilon = g + \varepsilon P$ has nongenerate roots at the points $a_1, \ldots, a_{k+1}$ for almost all values of ε.
(3°) For small $|\varepsilon|$ the index of the map $g_\varepsilon/\|g_\varepsilon\|$ of the boundary of U is equal to k.
(4°) Choose a small ε, for which the roots a_i of the map g are nondegenerate. Surround a_i by small balls B_i, not containing any other zeroes of the map g_ε. The degree of the map $g_\varepsilon/\|g_\varepsilon\|$ of the sphere ∂B_i to S_1^{2n-1} is equal to 1 and consequently the degree of the map $\cup \partial B_i$ to S_1^{2n-1} is equal to $k + 1$.

Consider the domain $U' = U \setminus \cup B_i$. The degree of the map of the boundary of this region is nonnegative (see the Remark above); on the other hand this degree is equal to $k - (k + 1) = -1$. Contradiction.

Corollary 1: *The index of a root is strictly positive.*

For the proof one has to apply the Lemma to a ball containing a single root of the system.

Corollary 2: *On the decomposition of an isolated root a finite number of roots are formed and the sum of their indices is equal to the index of the decomposed root.*

Corollary 3: *Under the conditions of Proposition* 3 *the index of each root does not exceed k.*

5.5 Multiplicity and A-equivalence

Proposition 1: *The multiplicities of A-equivalent germs are equal.*

In fact the ideals I_f and I_g of A-equivalent germs f and g coincide.

Proposition 2: *Suppose that a germ f has multiplicity μ and that the germ g differs from the germ f by small terms of order $\mu+1$. Then the germs g and f are A-equivalent.*

Corollary: *Suppose that the Jacobian matrix of the germ f at 0 is non-degenerate. Then its multiplicity is equal to 1.*

In fact this is clear for a linear map and a nonlinear map differs from a linear one by small terms of the second order.

Proposition 3: *A root of finite multiplicity of a system of holomorphic equations is isolated.*

For the proofs of Propositions 2 and 3 we require the

Lemma: *Let the germ f have multiplicity μ. Then the product of any μ function-germs, each taking the value 0 at 0, is contained in the ideal I_f.*

Proof of the Lemma: For the product $\varphi_1 \cdot \ldots \cdot \varphi_\mu$ we construct $\mu+1$ germs 1, φ_1, $\varphi_1\varphi_2$, $\ldots$, $\varphi_1 \cdot \ldots \cdot \varphi_\mu$. These germs are linearly dependent in the ring Q_f, that is there exist nontrivial linear combinations

$$c_0 + c_1\varphi_1 + \ldots + c_\mu\varphi_1 \cdot \ldots \cdot \varphi_\mu \in I_f.$$

Let c_r be the first coefficient different from zero; then

$$\varphi_1 \cdot \ldots \cdot \varphi_r(c_r + c_{r+1}\varphi_{r+1} + \ldots + c_\mu\varphi_{r+1} \cdot \ldots \cdot \varphi_\mu) \in I_f.$$

The multiplier within the brackets is invertible in the ring $\mathbb{C}\{x\}$, since $c_r \neq 0$. Consequently $\varphi_1 \cdot \ldots \cdot \varphi_r$ and therefore also $\varphi_1 \cdot \ldots \cdot \varphi_\mu$ belongs to the ideal I_f.

Proof of Proposition 2: Every function-germ φ of order $\mu + 1$ can be put in the form $\varphi = \Sigma h_i f_i$, where $h_i(0) = 0$ (using the Lemma). Having expressed all the components of $\varphi = g - f$ in this way we get $\varphi = Hf$, where $H(0) = 0$. Consequently, $g = (E + H)f$, which proves the A-equivalence of the germs f and g.

Proof of Proposition 3: Suppose that the germ f has multiplicity μ at 0. The germ x_j^μ we put in the form $x_j^\mu = \Sigma h_{j,i} f_i$. The region in which the germs $h_{j,i}$ and f_i may be holomorphically continued contains no roots of the system $f = 0$ other than the point 0.

5.6 Properties of the Pham map

Let f be a map-germ of multiplicity μ at 0. Consider the Pham map Φ^m, $m = \mu + 1, \ldots, \mu + 1$ and its deformation $\Phi_\varepsilon^m = \Phi^m + \varepsilon f$.

Proposition 1: *The germ f is A-equivalent at 0 to the germ Φ_ε^m for all $\varepsilon \neq 0$.*

Proof: The germ εf is A-equivalent to f at zero while Φ_ε^m differs from εf by small terms of order $\mu + 1$.

Proposition 2: *The index and the multiplicity at zero of the Pham map are equal to each other.*

Proof: (1°) The index is equal to the number of solutions of the system of equations $x_1^{m_1} = \varepsilon_1, \ldots, x_n^{m_n} = \varepsilon_n$ for general $\varepsilon_1, \ldots, \varepsilon_n$ (by Proposition 2 of Section 5.4). Consequently $\mathrm{ind}_0[\Phi^m] = m_1 \cdot \ldots \cdot m_n$.
(2°) The local algebra $Q_{\Phi^m,0}$ is generated by the monomials $x_1^{k_1} \cdot \ldots \cdot x_n^{k_n}$, where $0 \leqq k_1 < m_1, \ldots, 0 \leqq k_n < m_n$. The dimension $\mu_0[\Phi^m]$ of this algebra, consequently, is equal to $m_1 \cdot \ldots \cdot m_n$.

5.7 The subaddivity of multiplicity

Let $\{f_\varepsilon\}$ be an arbitrary deformation of a map-germ f of multiplicity μ at zero.

Proposition 1 (*on the subadditivity of the multiplicity*): *There is a neighbourhood of zero U in the source space such that for any sufficiently small $|\varepsilon|$ the number of roots of the system $f_\varepsilon = 0$ in U, counted with their multiplicities, does not exceed μ.*

Remark: The multiplicity is subadditive even in the real case, where, unlike the complex case, it is not additive.

Corollary: *The index of a germ of finite multiplicity is not greater than its multiplicity.*

For the proof of the corollary it is sufficient to apply the assertion concerning the subadditivity of the multiplicity to the particular deformation $f_\varepsilon = f - \varepsilon$.

Let $U \subset \mathbb{C}^n$ be an open set, $A(U)$ the algebra of holomorphic functions, defined on the set U and $I_g(U)$ the ideal of this algebra, generated by functions $g_1, \ldots, g_n$. The quotient algebra $Q_g(U) = A(U)/I_g(U)$ is said to be the *algebra of the map g on the domain U*.

The *polynomial subalgebra $Q_g[U]$ of the map g on the domain U* is defined to be the image of the algebra of polynomials in the algebra $Q_g(U)$ under the factoring homomorphism.

The subadditivity of the algebraic multiplicity follows from the following two propositions.

Proposition 2: *For every deformation $\{f_\varepsilon\}$ of a map-germ f of multiplicity μ at 0 there is a neighbourhood U of zero in the source space such that for any sufficiently small $|\varepsilon|$ the $\mathbb{C}$-dimension of the polynomial subalgebra of f_ε on U does not exceed μ.*

Proposition 3: *The number of solutions in U of the system of holomorphic equations $g = 0$, taking multiplicities into account, does not exceed the $\mathbb{C}$-dimension of the polynomial subalgebra of g on U.*

Proposition 3 is proved in Section 5.8. For the proof of Proposition 3 we require an addendum to the Weierstrass Preparation Theorem. Let f be a map-germ of finite multiplicity and let $e_1, \ldots, e_\mu$ be functions providing a basis for its local algebra. According to the Preparation Theorem there is for an arbitrary holomorphic map-germ a Weierstrass decomposition:

$$\varphi(x) = \Sigma e_i(x)\varphi_i(y), \quad y = f(x).$$

Lemma 1: *There exist single neighbourhoods of zero U_1 and U_2 in the target and source spaces on which the functions figuring in the Weierstrass decompositions of all the polynomials are simultaneously defined.*

Proof: For the domain U_1 we take the domain on which one can holomorphically continue the functions φ_i, participating in the decompositions of the following finite set of functions:

$$1,\ x_j e_k (1 \leqq j \leqq n, \quad 1 \leqq k \leqq \mu).$$

For the domain U_2 we take the subdomain of the domain $f^{-1}(U_1)$, on which the functions e_k can be holomorphically continued. We proceed by induction on polynomial degree. Every polynomial P of degree p can be put in the form

$$P = \Sigma x_j Q_j + c \cdot 1, \quad \deg Q_j < p.$$

We put in this representation the Weierstrass decompositions for the Q_j and use the Weierstrass decompositions of the functions $x_j e_k$ and 1. We get the decomposition of Lemma 1.

Consider the deformation $\{f_\varepsilon\}$, $\varepsilon \in \mathbb{C}^k$, of the holomorphic map-germ $f:(\mathbb{C}^n, 0) \to \mathbb{C}^n$. Define the map-germ $F:(\mathbb{C}^n \times \mathbb{C}^k, 0) \to \mathbb{C}^n \times \mathbb{C}^k$ by the formula $F(x, \varepsilon) = (f_\varepsilon(x), \varepsilon)$.

Lemma 2: *The local algebras of the germs f and F are isomorphic. If the functions $e_1, \ldots, e_\mu$ form a basis for the algebra of the germ f then they also form a basis for the algebra of the germ F.*

Proof: The ideal generated by the components $F_1, \ldots, F_n, \varepsilon_1, \ldots, \varepsilon_k$ of the map

F in the algebra of holomorphic map-germs at $0 \in \mathbb{C}^n \times \mathbb{C}^k$ coincides with the ideal generated by the functions $f_1, \ldots, f_n, \varepsilon_1, \ldots, \varepsilon_k$.

Let $e_1, \ldots, e_\mu$ be functions whose germs at 0 form a basis for the local algebra of the germ f and let $\{f_\varepsilon\}$ be a deformation of the germ f.

Lemma 3: *There is a neighbourhood of zero $U \subset \mathbb{C}^n$ such that for all sufficiently small $|\varepsilon|$ the linear envelope of the images of the functions $e_1, \ldots, e_\mu$ in the algebra $Q_{f_\varepsilon}(U)$ contains the polynomial subalgebra $Q_{f_\varepsilon}[U]$.*

Proof: The functions $e_1, \ldots, e_\mu$ form a basis for the local algebra of the map F (Lemma 2).

Apply Lemma 1 to the map F. According to this Lemma there is a neighbourhood of zero $U \times V \subset \mathbb{C}^n \times \mathbb{C}^k$ and a ball B in the target space $\mathbb{C}^{n+k}$ such that:
(1) $F(U \times V) \subset B$,
(2) in the domain $U \times V$ each polynomial P is representable in the form

$$(*) \qquad P(x) = \sum \Phi_i(y, \varepsilon) e_i(x), \quad y = f_\varepsilon(x).$$

By Hadamard's Lemma the functions Φ_i in B are representable in the form

$$\Phi_i(y, \varepsilon) = c_i(\varepsilon) + \sum_l y_l \Phi_{i,l}(y, \varepsilon).$$

Substitute these decompositions in (*). We get for each polynomial P in $U \times V$ a representation

$$(**) \qquad P(x) = \sum c_i(\varepsilon) e_i(x) + \sum h_j(x, \varepsilon) y_j, \quad y_j = f_{\varepsilon,j}(x),$$

where h_j is holomorphic on $U \times V$.

The second sum belongs to the ideal $I_{f_\varepsilon}(U)$. The Lemma is proved.

Remark: The linear combination of the functions e_i that we have constructed equivalent to the polynomial P modulo the ideal depends holomorphically on the parameter ε.

Proposition 2 follows from Lemma 3.

5.8 The estimate of the number of solutions of a system of equations.

In this section Proposition 3 of Section 5.7. is proved.

Lemma 1: *Suppose that the $\mathbb{C}$-dimension of the polynomial subalgebra of the map g in U is finite. Then every zero of the map g is of finite multiplicity.*

Proof: Suppose that a is a zero of the map g. Let φ_1 be linear functions taking the value 0 at a. If the dimension of the polynomial algebra is equal to μ, then the images in it of the $\mu+1$ polynomials $1, \varphi_1, \varphi_1 \cdot \varphi_2, \ldots, \varphi_1 \cdot \ldots \cdot \varphi_\mu$ are linearly dependent. Arguing as in the Lemma of Section 5.5 we find that there exists a function $\rho \in A(U)$ such that $\rho(a) \neq 0$ and $\rho\varphi_1 \cdot \ldots \cdot \varphi_\mu \in I_g(U)$. Inverting ρ in the algebra of holomorphic function-germs (but not polynomials) at a we get that $\varphi_1 \cdot \ldots \cdot \varphi_\mu \in I_{g,a}$. The lemma is proved.

Lemma 2: *The number of different roots of the system $g=0$ in U (without taking multiplicities into account) does not exceed the $\mathbb{C}$-dimension μ of the polynomial subalgebra of the map g in U.*

Proof: Let us suppose that there exist $\mu+1$ roots $a_1, \ldots, a_{\mu+1}$. There exist polynomials P_i, equal to 1 at a_i and equal to zero at the remaining μ roots. The images of the $\mu+1$ polynomials P_i in the polynomial subalgebra are linearly independent. This contradicts the condition.

Some notations. Let $a_1, \ldots, a_\nu$ be all the zeros of the map g in the domain U.

Definition: The *multilocal algebra* of the system $g=0$ in U is the direct sum of the local algebras of the germs of g at the points a_i.

Notation: $\Lambda_g(U) = \sum_{i=1}^{v} Q_{g,a_i}$.

We associate to every function of $A(U)$ the set of its germs at the points a_i. This association induces a homomorphism of the $\mathbb{C}$-algebra $A(U)$ to $\Lambda_g(U)$, which we shall denote by π.

Lemma 3: *Suppose that the $\mathbb{C}$-dimension of the polynomial subalgebra of the map g in U is finite. Then the image of the algebra of polynomials under the homomorphism π coincides with the multilocal algebra $\Lambda_g(U)$.*

Proof: Let $a_1, \ldots, a_\nu$ be the roots of the map g in the domain U (there are a finite number of them according to Lemma 2).

Each root a_i has finite multiplicity μ_i (by Lemma 1). Functions whose jets of order μ_i at a_i coincide determine the same elements in the local algebra Q_{g,a_i}. There exists a polynomial with arbitrary prescribed jets of orders μ_i at the finite set of points $a_1, \ldots, a_\nu$.

Proposition 3 follows from Lemma 3, since the ideal $I_g(U)$ is mapped to zero by the homomorphism π.

5.9 Isolatedness and finite multiplicity

We prove Theorem 2 of Section 5.2. It has been proved that a root of finite multiplicity of a system of holomorphic equations is isolated (see Section 5.5). It remains to prove the

Proposition: *An isolated root is of finite multiplicity.*

Proof: Let 0 be an isolated root of the system $f=0$. According to the local variant of the Hilbert zero theorem there is a number N such that $x_j^N \in I_{f,0}$. The proposition is proved.

We give now a direct proof, not using the zero theorem.

Suppose that B is a ball in the domain of convergence of the Taylor series of the germ of f at 0 and that the system $f=0$ has a single root 0 in B.

Lemma: *For each k there is a polynomial map g such that:*
(1) *the jets of f and g of order k at 0 are equal,*
(2) *the germ of g at 0 is of finite multiplicity,*
(3) $\|f\| > \|f-g\|$ *on the sphere ∂B.*

Proof: Put g in the form $g = f_{l-1} + \varepsilon x^l$, where f_{l-1} is the Taylor polynomial of f of degree $l - 1 > k$ and x^l is the Pham map Φ^m, $m = l, \ldots, l$.
(1°) The germ of g is of finite multiplicity at zero. Indeed, in the polynomial subalgebra of g in $\mathbb{C}^n$ the relations $\varepsilon x^l = -f_{l-1}$ hold true. Using these one can lower the degree of each polynomial, if its degree in one of the variables is greater than or equal to l. Consequently the dimension of the polynomial subalgebra is finite and every zero of g is of finite multiplicity (see Section 5.8.).
(2°) Choose l and then ε such that $\|f\| > \|f - g\|$ on the sphere ∂B. The Lemma is proved.

Proof of the Proposition: Choose a map g for $k = \mathrm{ind}_0[f]$.
(1°) The degree of the map $g/\|g\|$ of the sphere ∂B to the unit sphere is equal to k (condition 3).
(2°) $\mathrm{ind}_0[g] \leqq k$ (Corollary 3 of Section 5.4).
(3°) $\mu_0[g] = \mathrm{ind}_0[g] \leqq k$ (Theorem 1 of Section 5.2).
(4°) The germs of f and g are A-equivalent at zero, since they differ by small terms of order $k + 1$ or higher (Proposition 2 of Sections 5.5). Consequently the germ of f at 0 is of finite multiplicity.

5.10 The multilocal algebra of a decomposing root

In Sections 5.3 to 5.8 were verified all the propositions used in Section 5.2 in the proof of Theorem 1. These propositions also contain additional information.

Suppose that L is a $\mathbb{C}$-linear space, spanned by functions $e_1, \ldots, e_\mu$, whose germs at zero form a basis for the local algebra of the map f.

Theorem: *For each deformation $\{f_\varepsilon\}$ of the germ of f there is a neighbourhood of zero $U \subset \mathbb{C}^n$ and a neighbourhood of zero V in the parameter space such that for any $\varepsilon \in V$*
(1) *the map $\pi: L \to \Lambda_{f_\varepsilon}(U)$ is an isomorphism of linear spaces;*
(2) *each polynomial P in the algebra $A(U)$ is equivalent modulo the ideal $I_f(U)$ to a unique element of the space L and this element depends analytically on ε.*

Proof: (1) follows from the fact that the map $\pi: L \to \Lambda_{f_\varepsilon}(U)$ of spaces of the same dimension is surjective (since the polynomial subalgebra maps "onto", every polynomial is congruent to an element of L). The uniqueness in (2) follows from (1) and the holomorphicity from the Remark of Section 5.7.

Problem: The isomorphism $\pi: L \to \Lambda_{f_\varepsilon}(U)$ gives the linear space L the structure of an algebra, depending on the parameter ε. Show that this structure depends holomorphically on ε (that is that the product of two elements of L depends holomorphically on ε).

5.11 Bilinear forms on the local algebra

Suppose that $f:(\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is a map-germ of multiplicity $\mu < \infty$ and that Q_f is its local algebra. We define on Q_f a family of symmetric bilinear forms and prove their nondegeneracy.

Consider the Jacobian $J = \det(\partial f/\partial x)$, computed in some system of coordinates. We shall also denote by J the class of the Jacobian in Q_f and call it the Jacobian.

Theorem 1: *The Jacobian does not belong to the ideal I_f.*

Consider any linear form $\alpha: Q_f \to \mathbb{C}$. We define a bilinear form B_α on Q_f by the formula

$$B_\alpha(g, h) = \alpha(g \cdot h).$$

Theorem 2: *The bilinear form B_α is nondegenerate if and only if $\alpha(J) \neq 0$.*

The *annihilator* (*ann* I) of an ideal I is the set of all g such that $gi = 0$ for all i in I. The annihilator of an ideal is an ideal.

Corollary 1: *If $\alpha(J) \neq 0$ then the annihilator of an ideal in Q_f coincides with its orthogonal complement with respect to the form B_α.*

Proof: (1°) If $ai = 0$ then $B_\alpha(a, i) = 0$.
(2°) If $B_\alpha(a, i) = 0$ for all i in I but $ai_0 \neq 0$, then by the nondegeneracy of B_α

there is an element c for which $B_\alpha(ai_0, c) \neq 0$. But $B_\alpha(ai_0, c) = B_\alpha(a, i_0c) = 0$, since $i_0c \in I$.

Corollary 2: $ann(ann\, I) = I$.

Proof: $(I^\perp)^\perp = I$.

The proof of Theorems 1 and 2 is based on the construction of a special form $B = B_{\alpha_0}$.

Consider the algebra Q of functions on the μ points a_i. Take the linear form l on Q, $l(h) = \Sigma\varphi(a_i)h(a_i)$, constructed with respect to the "weight function" φ. Define the bilinear form $B(h, g)$ on Q by the formula $B(h, g) = l(h \cdot g)$. This form is nondegenerate if the weight function does not reduce to zero at any of the points a_i.

The local algebra Q_f is the algebra of functions on μ coincident points. It can be shown that it is possible to choose φ in such a way that for coincidence of the points the form B on Q has a well-defined limit and is moreover a non-degenerate form on Q_f. For this φ must tend to infinity on coincidence of the points (for otherwise the limit form would be degenerate). It can be shown that it is sufficient to take $\varphi = 1/J$, where J is the Jacobian of f.

The root 0 of the system $f = 0$ decomposes into the μ roots of the system $f = \varepsilon$ for small regular values ε. Let $a_1, \ldots, a_\mu$ be these roots. For any holomorphic function h at 0 we set

$$l^\varepsilon(h) = \Sigma h(a_i)/J(a_i).$$

Proposition 1: *As the regular value ε tends to zero $l^\varepsilon(h)$ tends to a finite value.*

We shall denote this limit by the symbol $[h/f]$.

Example 1: For the function $h = gJ$ the equality $[h/f] = \mu g(0)$ holds.

Proposition 2: *The linear form $\alpha_0(\cdot) = [\cdot/f]$ is equal to zero on the ideal I_f and consequently determines a linear form on the local algebra Q_f.*

Proposition 3: *The bilinear form $B = B_{\alpha_0}$ on the local algebra, constructed from the linear form $\alpha_0(\cdot) = [\cdot/f]$ is nondegenerate.*

The proof of Propositions 1–3 is given in Sections 5.14–5.18. We derive Theorems 1 and 2 from them.

Proof of Theorem 1: $[J/f] = \mu \neq 0$. Consequently, $J \notin I_f$ (Proposition 2).

Proof of Theorem 2: Any linear form α on Q_f has the form $\alpha(\cdot) = B(\cdot, \alpha^*)$ (since the form B is nondegenerate). Therefore $B_\alpha(h, g) = B(h, g\alpha^*)$. The form $B(h, g\alpha^*)$ is nondegenerate if and only if the element α^* is invertible, but $\alpha(J) = B(J, \alpha^*) = \mu\alpha^*(0)$ (Example 1). Therefore α^* is invertible if and only if $\alpha(J) \neq 0$.

Corollary 3: *The ideal generated by the Jacobian in Q_f is one-dimensional and does not depend on the system of coordinates used in the definition of the Jacobian. This ideal is contained in any nonzero ideal of the algebra Q_f.*

Proof: The equality in Example 1 shows that the maximal ideal $\boldsymbol{m}$ *is the* B-orthogonal complement to the line λJ. This line is therefore an invariantly defined ideal – the annihilator of the maximal ideal (Corollary 1). For a nonzero ideal I the inclusion $I^\perp \subseteq \boldsymbol{m}$ holds and consequently the inclusion $m^\perp \subseteq I$.

Remark: The symbol $[h/f]$ admits the *integral representation*

$$[h/f] = \left(\frac{1}{2\pi i}\right)^n \int \frac{h\, dx_1 \wedge \ldots \wedge dx_n}{f_1 \cdot \ldots \cdot f_n},$$

where the integration is along the small cycle, given by the equations $|f_k|^2 = \delta_k$ (see Section 5.18). One can take this formula as the defintion of the symbol and, starting from it, prove the properties of the symbol and with them also Theorems 1 and 2.

5.12 The index of a singular point of a real germ

Let $f:(\mathbb{R}^n, 0)\to(\mathbb{R}^n, 0)$ be a real-analytic map of multiplicity $\mu<\infty$ and let Q_f be its local $\mathbb{R}$-algebra. Choose orientations in both $\mathbb{R}^n$'s and denote by J the Jacobian, computed with respect to these oriented coordinates.

Consider any form $\alpha: Q_f\to\mathbb{R}$. Define the bilinear form B_α on Q_f by the formula $B_\alpha(g, h)=\alpha(g\cdot h)$.

Theorem (*the signature formula*): *The signature of the bilinear form B_α is equal to the index of the singular point* 0 *of the germ f if $\alpha(J)>0$.*

The proof is obtained by a limit procedure from the proposition given below concerning functions on a finite set with an involution.

A complex function on a set with involution τ is said to be τ-real if $\varphi(\tau a)=\overline{\varphi(a)}$ (a polynomial with real coefficients is τ-real for the involution of complex conjugation). All the τ-real functions on a set of μ points form an $\mathbb{R}$-algebra R of $\mathbb{R}$-dimension μ. For each function $\varphi\in R$ we define a bilinear form B_φ on R by the formula $B_\varphi(h, g)=\Sigma\varphi(a_i)h(a_i)g(a_i)$. Suppose that φ does not vanish at any of the points a_i.

Proposition 1: (1) *The values of the form B_φ are real.* (2) *The form B_φ is nondegenerate.* (3) *The signature of the form B_φ is equal to $\varphi^+-\varphi^-$, where φ^+ is the number of fixed points of the involution on which $\varphi>0$, and φ^- is the number on which $\varphi<0$.*

Proof: Under the action of the involution the set decomposes into invariant subsets, consisting of one or two points. Therefore it is sufficient to prove the proposition for one point and two point sets, for which it can be verified immediately.

We prove the signature formula for the special bilinear form B. The root 0 of the system $f=0$ breaks up for small real regular values ε into the μ complex roots of the system $f=\varepsilon$. Let $a_1, \ldots, a_\mu$ be these roots. The involution of complex conjugation acts on the set of these roots. We fix μ real polynomials $e_1,\ldots,e_\mu$ determining an $\mathbb{R}$-basis for the local algebra $\mathbb{R}\{x\}/(f)$ and, consequently, a $\mathbb{C}$-basis for the algebra $\mathbb{C}\{x\}/(f)$. Denote the spaces of their $\mathbb{R}$-linear and

$\mathbb{C}$-linear combinations by $L_{\mathbb{R}}$ and L. Consider the bilinear form B^{ε} on the space $L_{\mathbb{R}}$ defined by the formula

$$B^{\varepsilon}(g, h) = \sum \frac{g(a_i)h(a_i)}{J(a_i)}.$$

Lemma 1: *The signature of the form B^{ε} is equal to the number of real roots of the system $f = \varepsilon$, counted with the signs of the Jacobians at a_i.*

Corollary: *The signature of the form B^{ε} is equal to the index at zero of the map f (see Proposition* 1 *of Section* 5.3).

Lemma 1 follows from Proposition 1 and the following lemma.

Lemma 2: *The restrictions of the functions of $L_{\mathbb{R}}$ on the set of complex roots $(a_1, \ldots, a_\mu)$, and only these, are τ-real for the involution τ of complex conjugation.*

Proof: The τ-reality of the restrictions is obvious. Therefore it is enough to prove that the map of the μ-dimensional space $L_{\mathbb{R}}$ to the μ-dimensional space of τ-real functions does not have a kernel. But for the restriction map of functions on L to the set $(a_1, \ldots, a_\mu)$ only zero maps to zero (Section 5.10). Lemma 2 is therefore proved.

Let ε tend to zero. The form B^{ε} will then tend to a well-defined form B, corresponding to the linear form $\alpha_0(\cdot) = [\cdot/f]$ (Propositions 1 and 2 of Section 5.11). The limit form B is nondegenerate, since its complexification is nondegenerate (Proposition 3 of Section 5.11). Consequently its signature, like the signature of the pre-limit form B^{ε}, is equal to the index of the germ f at zero. Thus the signature formula has been proved for the special linear form α_0 (notice that $\alpha_0(J) = \mu > 0$). Now let α be an arbitrary linear form on the local $\mathbb{R}$-algebra, positive on the Jacobian. Join α and α_0 by a segment in the half-space of linear forms positive on the Jacobian. To the points of the segment there correspond nondegenerate bilinear forms (Theorem 2 of Section 5.11). Therefore their signatures are the same.

Remark: In [23] the signature formula is used to estimate the index of a singular point of a homogeneous vector field in $\mathbb{R}^n$ in terms of the degrees of the components of the field. In [100] the signature formula of Proposition 1 is used to estimate the total index of the singular points of a polynomial field in a domain of $\mathbb{R}^n$, defined by a polynomial inequality $P > 0$, in terms of the degrees of the components of the field and the polynomial P (the signature formula is applied in just the same way as in Lemma 2). The estimates are sharp. They generalise the well-known inequalities of Petrovskiĭ-Oleĭnik [140] in real algebraic geometry.

5.13 The inverse Jacobian theorem

Let $U \subset \mathbb{C}^n$ be a bounded domain with boundary and let $f: U \to \mathbb{C}^n$ be a holomorphic map. Let us assume that the system $f = 0$ has roots in U and that the image of the boundary $f(\partial U)$ does not contain 0. Let V be the connected component of 0 in $\mathbb{C}^n \backslash f(\partial U)$. The number of roots of the system $f - y = 0$ in U, taking multiplicities into account, is the same for all y in V (this follows from Proposition 3 of Section 5.4). Let $J = \det(\partial f/\partial x)$ and let h be a holomorphic function on U.

Theorem (*concerning the inverse Jacobian*): *On V there is a (unique) holomorphic function φ such that for any regular value y, $\varphi(y) = \Sigma h(a_i)/J(a_i)$, where the summation is over the set of all the roots a_i of the system $f - y = 0$ in U.*

A proof of the theorem, based on an n-dimensional version of Abel's theorem on the trace, is given in Section 5.18. We use this theorem straight away.

Suppose that the map f has a single zero in the ball B at its centre a and that the function h is holomorphic in B.

Corollary 1: *Let a_i be the roots of the system $f = \varepsilon$ in B. As the regular value ε tends to zero the function*

$$\varphi(\varepsilon) = \Sigma h(a_i)/J(a_i)$$

has a limit.

Definition: The limit in Corollary 1 is called the *symbol* $[h/f]_a$.

Let $\{f_\varepsilon\}$ be a deformation of the map f and $\{h_\varepsilon\}$ a deformation of the function h.

Corollary 2: *Let ε tend to zero in such a way that all the roots a_i of the system $f_\varepsilon = 0$ in B remain nondegenerate. Then*

$$\lim_{\varepsilon \to 0} \Sigma h(a_i)/\det(\partial f/\partial x)(a_i) = [h/f]_a.$$

The proof is obtained by applying Corollary 1 to the map $F: \mathbb{C}^n \times \mathbb{C}^k \to \mathbb{C}^n \times \mathbb{C}^k$ and the function H defined by the formulas $F(x, \varepsilon) = (f_\varepsilon(x), \varepsilon)$ and $H(x, \varepsilon) = h_\varepsilon(x)$.

We return to the situation of the inverse Jacobian theorem.

Corollary 3: *The function $\varphi(y) = \Sigma[h/f]_{a_i}$ is analytic on V. Here the summation is over the set of all roots of the system $f - y = 0$ in U.*

Proof: Take a regular value of the map close to y. The preimages of this value fall into groups lying close to the roots a_i. Let the regular value tend to y. On proceeding to the limit in each of the groups we get that the holomorphic function in the theorem is $\Sigma[h/f]_{a_i}$.

The *Euler–Jacobi formula* follows from the inverse Jacobian theorem. Let f be a polynomial map of $\mathbb{C}^n$ to $\mathbb{C}^n$ whose components are polynomials of degreees m_i. Let $f_0: \mathbb{C}^n \to \mathbb{C}^n$ be the polynomial map, whose components are the highest homogeneous terms of the components of f. Suppose that all of the roots a_i of the system $f = 0$ are simple and suppose that the system $f_0 = 0$ has a single root, the point 0.

Corollary 4 (the Euler–Jacobi formula): *For any polynomial h of degree less than the degree of the Jacobian* ($\deg h < m_1 + \cdots + m_n - n$),

$$\Sigma h(a_i)/J(a_i) = 0.$$

Proof: Consider $\mathbb{C}^n$ to be the coordinate plane $x_{n+1} = 0$ in $\mathbb{C}^{n+1}$. Let $\tilde{f}_i$ and $\tilde{h}$ be homogeneous polynomials in $\mathbb{C}^{n+1}$ such that $\tilde{f}_i(x, 1) = f_i(x)$, $\tilde{h}(x, 1) = h(x)$ and

$\deg\tilde{f}_i = \deg f_i$, $\deg\tilde{h} = \deg h$. Consider the map P: $(\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{n+1}, 0)$ with components $P_i = \tilde{f}_i$, for $i = 1, \ldots, n$, and $P_{n+1} = x_{n+1}$. Then $0 \in \mathbb{C}^{n+1}$ is the only root of the system $P = 0$. The roots of the system $P = (0, \varepsilon)$ (where $(0, \varepsilon) \in \mathbb{C}^n \times \mathbb{C}^1$) are points of the form $b_i = (a_i\varepsilon, \varepsilon)$, where a_i is a root of the system $f = 0$. At every root b_i we have the equality $\tilde{h}(b_i)/\det(\partial P/\partial\tilde{x})(b_i) = \varepsilon^p h(a_i)/J(a_i)$, where $p = \deg h - (m_1 + \cdots + m_n - n)$ ($\tilde{x}$ denotes the coordinates $x_1, \ldots, x_{n+1}$ in $\mathbb{C}^{n+1}$). Summing over all the roots we get

$$\Sigma\tilde{h}(b_i)/\det(\partial P/\partial x)(b_i) = \varepsilon^p \Sigma h(a_i)/J(a_i).$$

According to Corollary 1 the sum on the left must have finite limit as $\varepsilon \to 0$. For $p < 0$ this is possible only if $\Sigma h(a_i)/J(a_i) = 0$.

Remark 1: The Euler–Jacobi formula remains true if in place of the polynomials h and f with fixed degrees we consider polynomials with fixed quasidegrees. Other generalisations of the Euler–Jacobi formula are to be found in [85] and [99].

Remark 2: The Euler–Jacobi formula partially explains the existence of the limit in Corollary 1. Let f be a polynomial map and let h be a polynomial for which $\deg h < \deg J$. Suppose that the system $f = 0$ has exactly one multiple root 0 and some simple roots. Suppose also that in this system there are no roots "infinitely far away". Then as the regular value ε tends to zero some of the roots will tend to 0 and the remainder to the simple roots b_i. In this case it follows from the Euler–Jacobi formula that the limit of Corollary 1 exists and is equal to $-\Sigma h(b_i)/J(b_i)$.

Remark 3: The Euler–Jacobi formula has applications in real algebraic geometry (see [140], [100]).

5.14 Properties of the symbol $[h/f]_a$

Proposition 1: *Suppose that the germs g_a and f_a are A-equivalent, $g(\cdot) = A(\cdot)f(\cdot)$. Then*

$$[h/f]_a = [h \cdot \det A/g]_a.$$

Proof: Consider the deformation $f - \varepsilon$ of the germ f and the deformation $g_\varepsilon = A(f - \varepsilon)$ of the germ g. They have the same zeros a_i in a small ball. At each zero a_i we have the equality $(\partial g_\varepsilon/\partial x)(a_i) = A(a_i)(\partial f/\partial x)(a_i)$. Therefore

$$\Sigma h(a_i)/\det(\partial f/\partial x)(a_i) = \Sigma h(a_i)\det A(a_i)/\det(\partial g_\varepsilon/\partial x)(a_i).$$

On letting the regular value ε tend to zero we get the required equality.

Proposition 2: *If $h \in I_{f,a}$, then $[h/f]_a = 0$.*

(1°) Suppose that in addition the differentials df_k of all the components f_k do not have zeros in a punctured neighbourhood of a. Let $h = \Sigma g_k f_k$. We shall show that for every k the symbol $[g_k f_k/f]_a = 0$. The hypersurface $f_k = 0$ has no singularities in the punctured neighbourhood of a (by hypothesis). For $\varepsilon_k = 0$ the roots a_i of the system $f_j = \varepsilon_j$ lie on the hypersurface $f_k = 0$. For general ε (under the condition $\varepsilon_k = 0$) all the roots are simple and each term of the sum $\Sigma(g_k f_k)(a_i)/J(a_i)$ is equal to zero. Proceeding to the limit we obtain the required equality.

(2°) Each germ g of finite multiplicity is A-equivalent to a germ f for which the additional hypothesis of 1° holds. For the proof it is enough to put $f_i = \tilde{g}_i + (x_1^N + \cdots + x_n^N)$, where $\tilde{g}_i$ is the Taylor polynomial of degree $N - 1$ of the component g_i (see 1° of the Lemma of Section 5.9) and N is sufficiently large $(N > \mu_a[g])$ (see Proposition 2 of Section 5.5).

(3°) Suppose that $g = Af$, f satisfies the hypothesis of 1° and $h \in I_{f,a}$. Then $[h/g]_a = [h \cdot \det A/f]_a = 0$, since $h \cdot \det A \in I_{g,a} = I_{f,a}$.

5.15 The nondegeneracy of the bilinear form

The symbol $[h/f]_a$ depends only on the image of h in the algebra $Q_{f,a}$ (Proposition 2 of Section 5.14) and consequently determines a linear function on the algebra $Q_{f,a}$. In this section we consider the bilinear form B on the local algebra of a germ of finite multiplicity constructed from this linear function.

Proposition 1: *On the decomposition of a root of finite multiplicity with a nondegenerate bilinear form only roots with nondegenerate forms arise.*

Proof: Let f be a germ of finite multiplicity at a and L the $\mathbb{C}$-linear space spanned by the functions $e_1, \ldots, e_\mu$, whose germs form a basis for the local algebra of the germ f. Let $\{f_\varepsilon\}$ be a deformation of f and U a sufficiently small neighbourhood of a. The natural projection $\pi: L \to \wedge_{f_\varepsilon}(U)$ of the space L to the multilocal algebra of the system $f_\varepsilon = 0$ is an isomorphism for small ε (the theorem of Section 5.10). Consider the bilinear form B^ε on L, defined by the formula

$$B^\varepsilon(g, h) = \Sigma[g \cdot h/f]_{a_i},$$

where the summation is taken over all the roots of the system $f_\varepsilon = 0$ in U. This form is the direct sum of the bilinear forms of the roots a_i. The matrix $A^\varepsilon = \{B^\varepsilon(e_i, e_j)\}$ of the form B^ε depends analytically on ε according to Corollary 3 (Section 5.13). By hypothesis the bilinear form of the germ f is nondegenerate, that is, $\det A^\circ \neq 0$. Consequently for small $|\varepsilon|$ $\det A^\varepsilon \neq 0$. For such ε the bilinear forms of all the roots are nondegenerate.

Proposition 2: *The bilinear form of the germ of the Pham map is nondegenerate.*

The proof is obtained from the following computations. The local algebra of the Pham map Φ^m is generated by the monomials $x^k = x_1^{k_1} \cdot \ldots \cdot x_n^{k_n}$, $0 \leqq k_1 < m_1, \ldots, 0 \leqq k_n < m_n$. The monomial x^r, where $r = m_1 - 1, \ldots, m_n - 1$, is proportional to the Jacobian of the Pham map. For this monomial $[x^r/\Phi^m] = 1$. For all other x^k of the local algebra $[x^k/\Phi^m] = 0$. This follows from the Euler–Jacobi formula. The bilinear form of the germ of the Pham map is nondegenerate: dual to the basis x^k for Q_{Φ^m} is the basis x^{r-k}.

Proposition 3: *The bilinear form of any germ of finite multiplicity is nondegenerate.*

Proof: A-equivalent germs have their bilinear forms either both degenerate or both nondegenerate (this follows from Proposition 1 of Section 5.14). Every germ of finite multiplicity up to A-equivalence can be obtained from the germ of a Pham map by a small deformation (Section 5.6). The bilinear form of a Pham map-germ is nondegenerate. Proposition 3 now follows from Proposition 1.

5.16 The trace theorem

Consider a map f of complex manifolds of the same dimension, for which every point has a finite number of preimages. Let ω be a k-form on the source manifold.

Definition: The *trace* of the k-form ω by the map f is the k-form on the target manifold, whose value on each k-vector is equal to the sum of the values of the form ω on all the preimages of this k-vector. This form is defined for regular values of f. It is denoted by $\mathrm{Tr}\omega$.

Theorem (Abel): *Let $f(x) = x^p$ and $\omega = g\,dx$, where g is a function holomorphic at* 0. *Then the form $Tr\omega$, defined in a punctured neighbourhood of* 0, *continues holomorphically over* 0.

Proof: $Tr\omega = \varphi dy$, where $\varphi(y) = \Sigma g(y^{1/p})(1/p)y^{(1/p)-1}$. Represent φ as a power series in $y^{1/2}$. The coefficients of the non-integral powers of y are equal to 0, since φ is single-valued. There are then no negative powers of y in the decomposition, since each term has degree not less than $(1/p) - 1$.

Corollary 1: *Let f be a one-dimensional ramified μ-fold covering. Then the trace of a holomorphic form on the source space continues holomorphically to a form on the target space.*

To formulate the trace theorem in the n-dimensional case we make the following

Definition: A map f of complex manifolds of the same dimension is said to be of *finite type* if the sum of the multiplicities of all the preimages of each point has a constant finite value. This value μ is said to be the *number of leaves* of the projection $f: M \to N$, a covering of finite type of M over N.

Proposition: *A map of finite type is proper.*

Proof: The point y has μ preimages (counting multiplicities). Every point sufficiently close to y has μ preimages (counting multiplicities) close to the preimages of y. Consequently there are no other preimages and this means that the map is proper.

Corollary 2: *The set of regular values of a map of finite type is open and everywhere dense.*

Theorem: *The trace of a holomorphic form for a map of finite type extends holomorphically over the whole target manifold.*

The proof may be obtained from Abel's theorem in the following way.
(1°) For the map $f(x_1, \ldots, x_n) = (x_1^p, x_2, \ldots, x_n)$ the theorem is proved just as for Abel's theorem.
(2°) A point of the source manifold is said to be *good* if there are systems of coordinates in neighbourhoods of the point and its image in which the map is described by the formula of (1°).

A point of the target manifold is said to be *good* if all its preimages are good.

In a neighbourhood of a good point in the target the theorem follows from (1°).
(3°) The set of bad points in the source has codimension greater than 1. For the proof let us consider the following three sets in the source:
(1) the set of singularities of the set of critical points of the map;
(2) the set of critical points of the restriction of f to the nonsingular part of the set of critical points;
(3) the set of critical points of f at which the multiplicity is greater than at neighbouring critical points.

It is not difficult to prove that the codimension of each of these sets is greater than 1 for maps of finite type, and all the remaining points are good.
(4°) The set of bad points in the source is analytic, hence the set of bad points in the target is analytic (Remmert's theorem) of codimension greater than 1.
(5°) By Hartog's theorem the trace extends holomorphically to the set of bad points.

Another proof of the trace theorem is given below, without reference to the theorems of Remmert and Hartogs.

5.17 The integral representation of the trace

Let $f: M \to V$ be a map of finite type onto a domain V of $\mathbb{C}^n$ and let ω be a holomorphic n-form on M. Choose in $\mathbb{C}^n$ coordinates $y_1, \ldots, y_n$. Define the function $[\mathrm{Tr}\omega]$ at regular values of the map as the coefficient in the representation

$$\mathrm{Tr}\omega = [\mathrm{Tr}\omega] dy_1 \wedge \ldots \wedge dy_n.$$

Consider the map $|f|^2 : M \to \mathbb{R}^n$, $x \mapsto (|f_1(x)|^2, \ldots, |f_n(x)|^2)$.

Let δ be a positive vector in $\mathbb{R}^n$. Define the polydisk V_δ by the conditions $|y_k|^2 < \delta_k$, and its shell T_δ by the conditions $|y_k|^2 = \delta_k$.

Theorem: *Let δ be a noncritical value of the map $|f|^2$ such that the polydisk V_δ together with its shell lies in V. Then in the polydisk the function* $[\mathrm{Tr}\omega]$ *admits the integral representation*

$$(*) \qquad [Tr\omega](y) = \frac{1}{(2\pi i)^n} \int_{\Gamma_\delta} \frac{\omega}{\Pi(f_i - y_i)},$$

where the cycle Γ_δ is defined by the condition $|f|^2 = \delta$.

We define a meromorphic n-form on M, depending on the point y of V, by the formula $\omega_y = \omega/[(2\pi i)^n \Pi(f_i - y_i)]$.

We define the map $|f - y|^2 : M \to \mathbb{R}^n$, depending on the parameter $y \in V$, by the formula $x \mapsto (|f_1(x) - y_1|^2, \ldots, |f_n(x) - y_n|^2)$. The preimage of a regular value $\rho \in (\mathbb{R}^+)^n$ is a compact manifold for sufficiently small y and ρ. Denote it by $\Gamma_{y,\rho}$.

Lemma 1: *In a neighbourhood of a regular value y of f*

$$[Tr\omega] = \int_{\Gamma_{y,\rho}} \omega_y$$

(for every sufficiently small vector ρ with positive components).

Proof: The preimage of a small neighbourhood W of a regular value consists of μ non-intersecting neighbourhoods U_j. We shall identify U_j with W by means of f and shall use in U_j the system of coordinates $f_1, \ldots, f_n$. In U_j let $\omega = g_j df_1 \wedge \ldots \wedge df_n$. Then $[Tr\omega] = \Sigma g_j$. On the other hand, for small ρ the cycle $\Gamma_{y,\rho}$ decomposes into μ tori T_j, lying in the neighbourhoods U_j (and defined in them by the n equations $|f_i - y_i|^2 = \rho_i$). According to the Cauchy integral formula

$$g_j(y) = \frac{1}{(2\pi i)^n} \int_{T_j} \frac{g_j(f_1, \ldots, f_n) df_1 \wedge \ldots \wedge df_n}{\Pi(f_i - y_i)}.$$

In our notations this equality takes the form $g_j(y) = \int_{T_j} \omega_y$. Consequently $\int_{\Gamma_{y,\rho}} \omega_y = \Sigma g_j(y)$. The Lemma is proved.

Lemma 2: *For every regular value of f sufficiently close to zero the cycle $\Gamma_{y,\rho}$ for small ρ ($|\rho| < \rho_0(y)$) is homologous to the cycle Γ_δ in the complement of the set of singularities of the form ω_y.*

The proof is based on the fact that the preimages of a regular value for two homotopic smooth proper maps are homologous.

(1°) The cycle Γ_δ is defined by means of a nondegenerate system of equations. A small change in these equations only changes the cycle slightly. For any y from a sufficiently small neighbourhood of zero W_0 the cycles $\Gamma(t) = \Gamma_{ty,\delta}$ form a smooth homotopy of Γ_δ to $\Gamma_{y,\delta}$ as t varies from 0 to 1. Analogously, for $y \in W_0$ and for any sufficiently small $\rho \in \mathbb{R}^n$ ($|\rho| < \rho_0(y)$) the cycles $\Gamma(t) = \Gamma_{y,\delta+t\rho}$ form a smooth homotopy of $\Gamma_{y,\delta}$ to $\Gamma_{y,\delta+\rho}$.

(2°) Consider the map $M \times \mathbb{R} \to \mathbb{R}^n$, depending on the parameter $y \in V$, sending the point (x, t) to the point with coordinates $|f_i(x) - y_i|^2 - t\delta_i$. Let ρ be a regular value of this map so small that $|\rho| < \rho_0(y)$ and that the cycle $\Gamma_{y,\rho}$ decomposes into μ tori T_j (see Lemma 1). The preimage of ρ defines a smooth $(n+1)$-dimensional submanifold in $M \times \mathbb{R}$. The projection to M of the part of this manifold distinguished by the inequalities $0 \leqq t \leqq 1$ is a film, stretched between the cycles $\Gamma_{y,\delta}$ and $\Gamma_{y,\rho} = \Sigma T_j$. The Lemma is proved, since neither the film nor the homotopies produced touch the singularities of the form ω_y.

Proof of the theorem: (1°) The representation (∗) holds for small y. In fact, by Lemmas 1 and 2 for small y

$$\int_{\Gamma_\delta} \omega_y = \int_{\Gamma_{y,\rho}} \omega_y = [\mathrm{Tr}\,\omega](y).$$

(2°) From (1°) there follows the trace theorem for n-forms.

(3°) From the trace theorem there follows the holomorphicity of the continuation of $[\mathrm{Tr}\,\omega]$ over the whole of V; The right hand side of (∗) is holomorphic in the polycylinder V_δ. According to (1°) the left and right hand sides coincide in a neighbourhood of zero; accordingly they coincide on V_δ.

Corollary: *There is an integral representation for the trace of a k-form on n-dimensional space.*

Consider, for example, the trace of a 1-form on $\mathbb{C}^2$, $\mathrm{Tr}\omega = a_1dy_1 + a_2dy_2$. Multiplying by dy_2, we get

$$a_1dy_1 \wedge dy_2 = (\mathrm{Tr}\omega) \wedge dy_2 = \mathrm{Tr}(\omega \wedge f^*dy_2)$$

and we get an integral representation for the coefficient a_1 as $[\mathrm{Tr}(\omega \wedge f^*dy_2)]$.

Analogously one finds integral representations for the coefficients of the coordinate form of the trace of a k-form in $\mathbb{C}^n$.

Remark: There is yet another proof of the trace theorem in [84]. There one can also read about applications and generalisations of this theorem.

5.18 Proof of the inverse Jacobian theorem

Consider the domain $f^{-1}(V) = M$ and its map $f|\,M\colon M \to V$. It is of finite type. Consider the n-form $\omega = h\,dx_1 \wedge \ldots \wedge dx_n$. On an open dense set in V of regular values of f, $[\mathrm{Tr}\omega] = \Sigma h(a_i)/J(a_i)$, and by the trace theorem the function $[\mathrm{Tr}\omega]$ is holomorphic in V. The theorem is proved.

Corollary:

$$[h/f] = \frac{1}{(2\pi i)^n}\int_{|f_i|=\delta_i} \frac{h\,dx_1 \wedge \ldots \wedge dx_n}{f_1 \cdot \ldots \cdot f_n}.$$

Proof: $[h/f] = \lim_{y\to 0}[\mathrm{Tr}\omega(y)] = [\mathrm{Tr}\omega](0)$, and therefore the corollary follows from the integral representation of the trace.